Lecture- 5

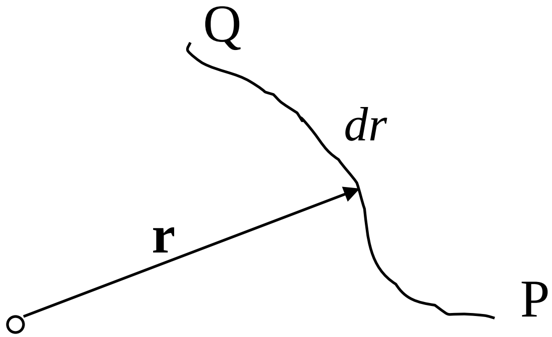
line , surface and volume integrals

Pradeep Singla

Vector integration

- Linear integrals
- Vector area and surface integrals
- Volume integrals

An arbitrary path of integration can be specified by defining a variable position vector **r** such that its end point sweeps out the curve between P and Q



A vector ***A*** can be integrated between two fixed points along the curve ***r*** :

$$\int_P^Q \mathbf{A} \cdot \mathbf{dr} = \int_P^Q (A_x dx + A_y dy + A_z dz)$$

Scalar product

If the integration depends on P and Q but not upon the path **r :**

$$\mathbf{A} \cdot \mathbf{dr} = d\phi = \nabla \phi \cdot dr$$

$$\therefore (\mathbf{A} - \nabla \phi) \cdot \mathbf{dr} = 0$$

if A·B = 0

The vector A is zero

The vector B is zero

$\theta = 90^\circ$

$$\mathbf{A} - \nabla \phi = 0$$

$$\mathbf{A} - \nabla\phi = 0$$

If a vector field **A** can be expressed as the gradient of a scalar field ϕ, the line integral of the vector **A** between any two points *P* and *Q* is independent of the path taken.

If ϕ is a single-valued function :

$$\frac{\partial}{\partial y}(A_x) = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial x}(A_y) \quad \text{and} \quad \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k} = 0$$

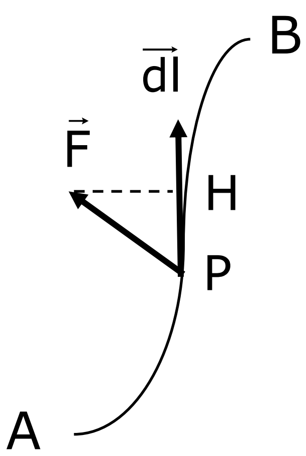
$$\nabla \times A = 0$$

$$A = \nabla\phi$$

$$\nabla \times A = 0$$

Example : $W = \int_P^Q F \cdot dr$

- **Conservative vector field**



Theorem. If there exists ϕ such that $\vec{\mathbf{F}} = \mathbf{grad}\,\phi$ then

$$\int_{AB} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dl}} = \phi(B) - \phi(A)$$

Consequently, the value of the integral doesn't depend on the path, but only on its beginning A and its end B. We say that the vectorfield $\vec{\mathbf{F}}$ is **conservative**

Proof. $$\int_{AB} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dl}} = \int_{AB} \mathbf{grad}\phi \cdot \overrightarrow{\mathbf{dl}} = \int_{AB} \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} \cdot dz =$$

$$= \int_{AB} d\phi = \phi(B) - \phi(A)$$

Work, force and displacement

If the vector field is a force field and a particle at a point **r** experiences a force **f**, then the work done in moving the particle a distance **δr** from **r** is defined as the displacement times the component of force opposing the displacement :

$$\delta W = F \cdot \delta r$$

The total work done in moving the particle from P to Q is the sum of the increments along the path. As the increments tends to zero:

$$W = \int_P^Q F \cdot dr$$

When this work done is independent of the path, the force field is "**conservative**". Such a force field can be represented by the gradient of a scalar function :

$$F = \nabla W$$

$$A = \nabla \phi$$

When a scalar point function is used to represent a vector field, it is called a "potential" function :

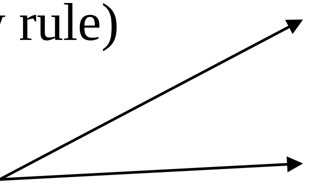
gravitational potential function (potential energy)……………….gravitational force field
electric potential function ………………………………………..electrostatic force field
magnetic potential function……………………………………….magnetic force field

Surface : a vector by reference to its boundary

area : the maximum projected area of the element

direction : normal to this plane of projection (right-hand screw rule)

$$\mathbf{dS} = \mathbf{n}\,dS$$



The surface integral is then :

$$\int \mathbf{A}\cdot\mathbf{dS} = \int \mathbf{A}\cdot\mathbf{n}\,dS$$

If **A** is a force field, the surface integral gives the total forace acting on the surface.

If **A** is the velocity vector, the surface integral gives the net volumetric flow across the surface.